Myopic Bosonization

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As the title suggests, this is an attempt at bosonizing fermions in any number of dimensions without paying attention to the fact that the Fermi surface is an extended object. One is tempted to introduce the density fluctuation and its conjugate and recast the interacting problem in terms of these canonical Bose fields. However, we find that the attempt is short-sighted figuratively as well for the same reason. But surprisingly, this flaw, which manifests itself as an inconsistency between Menikoff-Sharp's construction of the kinetic energy operator in terms of currents and densities, and our ansatz for this operator, is nevertheless able to reproduce(although reluctantly) many salient features of the free theory. Buoyed by this success, we solve the interacting problem and compute the full propagator.

I. DENSITY FLUCTUATION AND ITS CONJUGATE

We showed earlier [1] that the field operator, may be expressed in terms of the density fluctuation and its conjugate.

$$\psi(\mathbf{x}) = e^{-i\sum_{\mathbf{q}} e^{i\mathbf{q}.\mathbf{x}} X_{\mathbf{q}}} e^{i\sum_{\mathbf{q}} e^{-i\mathbf{q}.\mathbf{x}} \tilde{U}_0(\mathbf{q})\rho_{\mathbf{q}}/N} \sqrt{\rho_0} \quad (1)$$

where, $[X_{\mathbf{q}}, \rho_{\mathbf{q}'}] = i\delta_{\mathbf{q},\mathbf{q}'}$ and all other commutators are zero. Actually, this concept of a conjugate to the density fluctuation has appeared in many guises in the literature. Apparently, the first person to think of this in the context of Fermi systems was Pines [2]. Also,

$$\tilde{U}_0(\mathbf{q}) = (\frac{\theta(k_f - |\mathbf{q}|) - w_1(\mathbf{q})}{w_2(\mathbf{q})})^{\frac{1}{2}} \quad (2)$$

$$w_1(\mathbf{q}) = (\frac{1}{4\ N\ \epsilon_{\mathbf{q}}^2}) \sum_{\mathbf{k}} (\frac{\mathbf{k}.\mathbf{q}}{m})^2 (\Lambda_{\mathbf{k}}(-\mathbf{q}))^2 \quad (3)$$

$$w_2(\mathbf{q}) = (\frac{1}{N}) \sum_{\mathbf{k}} (\Lambda_{\mathbf{k}}(-\mathbf{q}))^2 \quad (4)$$

here, $\Lambda_{\mathbf{k}}(\mathbf{q}) = \sqrt{\bar{n}_{\mathbf{k}+\mathbf{q}/2}(1-\bar{n}_{\mathbf{k}-\mathbf{q}/2})}$ and $\bar{n}_{\mathbf{k}} = \theta(k_F - |\mathbf{k}|)$. Let us now postulate that the kinetic energy operator has the following form,

$$K = \sum_{\mathbf{k}} \omega_0(\mathbf{k}) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (5)$$

$\omega_0(\mathbf{k}) = \epsilon_{\mathbf{k}}/S(\mathbf{k})$ is the Bijl-Feynman dispersion [3]. Here $\epsilon_{\mathbf{k}} = k^2/2m$ and $S(\mathbf{q}) = q/2k_F$ is the static structure factor for small q. The fields $b_{\mathbf{k}}$ are canonical bosons. From this we may postulate,

$$X_{\mathbf{q}} = \frac{i}{2\sqrt{N\ S_{\mathbf{q}}} cos\theta_{\mathbf{q}}} (b_{-\mathbf{q}} e^{i\theta_{\mathbf{q}}} - b_{\mathbf{q}}^{\dagger} e^{-i\theta_{\mathbf{q}}}) \quad (6)$$

also,

$$\rho_{\mathbf{q}} = \sqrt{N\ S_{\mathbf{q}}} (b_{\mathbf{q}} + b_{-\mathbf{q}}^{\dagger}) \quad (7)$$

here $\theta_{\mathbf{q}}$ is a nontrivial phase. These cannot be absorbed into the $b_{\mathbf{q}}$'s, if they were they would pop up in the density fluctuation again. The current density fluctuation has a similar form,

$$\mathbf{j}_{\mathbf{q}} = \mathbf{q}\sqrt{\frac{N}{4S_{\mathbf{q}}}} (b_{\mathbf{q}} - b_{-\mathbf{q}}^{\dagger}) \quad (8)$$

This form reproduces the commutator between the density-fluctuation and the kinetic energy,

$$[\rho_{\mathbf{q}}, K] = \sqrt{N\ S_{\mathbf{q}}} \frac{\epsilon_{\mathbf{q}}}{S_{\mathbf{q}}} (b_{\mathbf{q}} - b_{-\mathbf{q}}^{\dagger}) = \mathbf{q}.\frac{\mathbf{j}_{\mathbf{q}}}{m} \quad (9)$$

However, from the field operator, we also have,

$$\mathbf{j}_{\mathbf{q}} = i\mathbf{q}N\ X_{-\mathbf{q}} - i\mathbf{q}\tilde{U}_0(\mathbf{q})\rho_{\mathbf{q}} \quad (10)$$

From these two we have,

$$tan\ \theta_{\mathbf{q}} = -2\ \tilde{U}_0(\mathbf{q})\ S_{\mathbf{q}} \quad (11)$$

$$\psi(\mathbf{x}) = e^{-\sum_{\mathbf{q}} e^{i\mathbf{q}.\mathbf{x}} \frac{b_{\mathbf{q}}^{\dagger}}{2\sqrt{NS_{\mathbf{q}}}}} e^{\sum_{\mathbf{q}} e^{i\mathbf{q}.\mathbf{x}} \frac{b_{-\mathbf{q}}}{2\sqrt{NS_{\mathbf{q}}}}}$$

$$\times\ e^{i\sum_{\mathbf{q}} \tilde{U}_0(\mathbf{q})/2N} e^{-\sum_{\mathbf{q}} \frac{1}{8NS_{\mathbf{q}}}} \sqrt{\rho_0} \quad (12)$$

The equal-time version of the propagator is,

$$\langle \psi^{\dagger}(\mathbf{x})\psi(\mathbf{x}^{'}) \rangle = \rho_0 e^{\sum_{\mathbf{q}} (e^{i\mathbf{q}.(\mathbf{x}-\mathbf{x}^{'})} - 1)\frac{1}{4NS_{\mathbf{q}}}} \quad (13)$$

but we also know that,

$$\langle \psi^{\dagger}(\mathbf{x})\psi(\mathbf{x}^{'}) \rangle = \frac{1}{V} \sum_{\mathbf{q}} e^{i\mathbf{q}.(\mathbf{x}-\mathbf{x}^{'})} \theta(k_f - |\mathbf{q}|)$$

$$= \rho_0 (1 + \frac{1}{N} \sum_{\mathbf{q} \neq 0} (e^{i\mathbf{q}.(\mathbf{x}-\mathbf{x}^{'})} - 1)\theta(k_f - |\mathbf{q}|)) \quad (14)$$

Therefore,

$$Ln(\langle \psi^{\dagger}(\mathbf{x})\psi(\mathbf{x}^{'})\rangle)$$

$$\approx ln(\rho_0) + \frac{1}{N} \sum_{\mathbf{q} \neq 0} (e^{i\mathbf{q}.(\mathbf{x}-\mathbf{x}^{'})} - 1)\theta(k_f - |\mathbf{q}|)) \quad (15)$$

Since

$$\frac{1}{4S_{\mathbf{q}}} \neq \theta(k_f - |\mathbf{q}|)) \quad (16)$$

the two propagators don't agree. But then we proceed since the title suggests that this is a short-sighted endeavour anyway. If we compute the full propagator and then multiply and divide by the free propagator and in the division use the form predicted by myopic bosonization and in the numerator use the form predicted by elementary considerations then, even though all is not well, we should be still alive at the end of the day, albeit battered and bruised. Let us introduce an interaction

$$H_I = \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{2V} (NS_{\mathbf{q}})(b_{\mathbf{q}} + b^{\dagger}_{-\mathbf{q}})(b_{-\mathbf{q}} + b^{\dagger}_{\mathbf{q}}) \quad (17)$$

From this we may diagonalise the full problem in terms of new Bose fields $d_{\mathbf{q}}$,

$$H = \sum_{\mathbf{q}} \omega_{\mathbf{q}} d^{\dagger}_{\mathbf{q}} d_{\mathbf{q}} \quad (18)$$

where,

$$\omega^2(\mathbf{q}) = \omega_0^2(\mathbf{q}) + 2\rho_0 v_{\mathbf{q}} \omega_0(\mathbf{q}) \quad (19)$$

and,

$$d_{\mathbf{q}} = [d_{\mathbf{q}}, b^{\dagger}_{\mathbf{q}}] b_{\mathbf{q}} - [d_{\mathbf{q}}, b_{-\mathbf{q}}] b^{\dagger}_{-\mathbf{q}} \quad (20)$$

and,

$$b_{\mathbf{q}} = [b_{\mathbf{q}}, d^{\dagger}_{\mathbf{q}}] d_{\mathbf{q}} - [b_{\mathbf{q}}, d_{-\mathbf{q}}] d^{\dagger}_{-\mathbf{q}} \quad (21)$$

$$[d_{\mathbf{k}}, b^{\dagger}_{\mathbf{k}}] = \frac{\sqrt{NS_{\mathbf{k}}}}{V} v_{\mathbf{k}} \frac{[d_{\mathbf{k}}, \rho_{-\mathbf{k}}]}{\omega(\mathbf{k}) - \omega_0(\mathbf{k})} \quad (22)$$

$$[d_{\mathbf{k}}, b_{-\mathbf{k}}] = -\frac{\sqrt{NS_{\mathbf{k}}}}{V} v_{\mathbf{k}} \frac{[d_{\mathbf{k}}, \rho_{-\mathbf{k}}]}{\omega(\mathbf{k}) + \omega_0(\mathbf{k})} \quad (23)$$

Since $[d_{\mathbf{k}}, d^{\dagger}_{\mathbf{k}}] = 1$,

$$[d_{\mathbf{k}}, \rho_{-\mathbf{k}}] = \sqrt{\frac{N}{S_{\mathbf{k}}}} \sqrt{\frac{\omega_0(k)}{\omega(\mathbf{k})}} \quad (24)$$

Let us write down formulas for the propagators,

$$\frac{\langle \psi^{\dagger}(\mathbf{x}^{'}t^{'})\psi(\mathbf{x}t)\rangle}{\langle \psi^{\dagger}(\mathbf{x}^{'}t^{'})\psi(\mathbf{x}t)\rangle_0} = e^{-\sum_{\mathbf{q}} \frac{1}{2NS_{\mathbf{q}}} [b_{\mathbf{q}}, d^{\dagger}_{\mathbf{q}}][b_{-\mathbf{q}}, d_{\mathbf{q}}]}$$

$$\times e^{-\sum_{\mathbf{q}} \frac{1}{2NS_{\mathbf{q}}} ([b_{\mathbf{q}}, d_{-\mathbf{q}}])^2}$$

$$\times e^{\sum_{\mathbf{q}} \frac{e^{i\mathbf{q}.(\mathbf{x}-\mathbf{x}^{'})}}{4NS_{\mathbf{q}}} \{ \frac{\omega_0(\mathbf{q})}{\omega(\mathbf{q})} e^{i\omega(\mathbf{q})(t-t^{'})} - e^{i\omega_0(\mathbf{q})(t-t^{'})} \}} \quad (25)$$

$$\frac{\langle \psi(\mathbf{x}t)\psi^{\dagger}(\mathbf{x}^{'}t^{'})\rangle}{\langle \psi(\mathbf{x}t)\psi^{\dagger}(\mathbf{x}^{'}t^{'})\rangle_0} = e^{-\sum_{\mathbf{q}} \frac{1}{2NS_{\mathbf{q}}} [b_{\mathbf{q}}, d^{\dagger}_{\mathbf{q}}][b_{-\mathbf{q}}, d_{\mathbf{q}}]}$$

$$\times e^{-\sum_{\mathbf{q}} \frac{1}{2NS_{\mathbf{q}}} ([b_{\mathbf{q}}, d_{-\mathbf{q}}])^2}$$

$$\times e^{\sum_{\mathbf{q}} \frac{e^{i\mathbf{q}.(\mathbf{x}^{'}-\mathbf{x})}}{4NS_{\mathbf{q}}} \{ \frac{\omega_0(\mathbf{q})}{\omega(\mathbf{q})} e^{i\omega(\mathbf{q})(t^{'}-t)} - e^{i\omega_0(\mathbf{q})(t^{'}-t)} \}} \quad (26)$$

As far as the dielectric function is concerned, it may be evaluated using methods outlined in our earlier work [1]. The final answer is as follows,

$$\epsilon(\mathbf{q}, \omega) = \frac{1}{1 + v_{\mathbf{q}} \rho_0 S_{\mathbf{q}} (\omega_{\mathbf{q}}/\omega^0_{\mathbf{q}}) (\frac{2\omega_{\mathbf{q}}}{\omega^2 - \omega^2_{\mathbf{q}}})} \quad (27)$$

Assuming that ω has a small imaginary part we recover damping as well. However, the above dielectric function apart from having obvious zeros at $\omega = \omega_{\mathbf{q}}$ does not seem to resemble the RPA dielectric function. This points to the fact that our approach is probably flawed. The main reason why our approach is wrong is because of the following reason. The Menikoff-Sharp construction [4] of the kinetic energy operator may be written as,

$$K = \int \frac{d\mathbf{x}}{2m} (\frac{\mathbf{J}^2}{\rho} + \frac{(\nabla\rho)^2}{4\rho}) + c - number \quad (28)$$

In momentum space it looks as follows(after expanding in powers of density fluctuations),

$$K = \frac{(\mathbf{j_0})^2}{2mN} + \sum_{\mathbf{q} \neq 0} \frac{\mathbf{j_q}.\mathbf{j_{-q}}}{2mN} + \sum_{\mathbf{q} \neq 0} \frac{\epsilon_{\mathbf{q}}}{4N} \rho_{\mathbf{q}} \rho_{-\mathbf{q}} \quad (29)$$

The current $\mathbf{j_0}$ is the total current operator $\mathbf{j_0} = \sum_{\mathbf{k}} \mathbf{k} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}$. This may be written as,

$$\mathbf{j_0} = -\sum_{\mathbf{q}} (i\mathbf{q}) \rho_{\mathbf{q}} X_{\mathbf{q}} \quad (30)$$

If one uses the formulas for the density fluctuation and its conjugate in terms of the Bosons $b_{\mathbf{q}}$ we find that the resulting kinetic energy operator is not the one written down in Eq.(5). The only purpose of this exercise is to point out to future scholars that this route is a dead end.

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